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Zeros of Spline Functions and Applications

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1. INTRODUCTION

It is known (cf. [6, 12]) that if s is a polynomial spline of order m with k simple knots, then $Z(s) \leqslant m + k - 1$, where Z counts the number of zeros of s with multiplicities. On the other hand, if s is a polynomial spline of order m with a total of K knots, counting multiple knots according to their multiplicity, then $Z(s) \leqslant m + K - 1$, where now Z counts each zero just once (see [19, 20]). (This result is also true if nonnodal zeros are counted twice.) The main purpose of this paper is to show that this bound on the number of zeros of a spline with *multiple* knots remains valid even if we take a very strong rule Z for counting *multiple* zeros.

The main result on zeros is proved in the following section. An analogous result for periodic splines can be found in Section 3. The remainder of the paper is devoted to some applications. In particular, we show how some rather complicated results on the determinants of certain matrices formed from B-splines or from a Green's function follow easily from the zero properties. We close the paper with a section giving remarks and further references.

2. ZEROS OF POLYNOMIAL SPLINES

We begin by defining the class of polynomial splines of interest. Let $a = x_0 < x_1 < \cdots < x_{k+1} = b$, and set $\Delta = \{x_i\}_1^k$. The set Δ partitions the interval $[a, b]$ into $k + 1$ pieces, $I_i = [x_i, x_{i+1})$, $i = 0, 1,\ldots, k - 1$ and $I_k = [x_k, x_{k+1}]$. Given a positive integer m, let $\mathscr{M} = (m_1, \ldots, m_k)$ be a vector

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of integers satisfying $1 \leqslant m_i \leqslant m$, $i = 1, 2, \ldots, k$. We define the space of *polynomial splines of order m with knots at* $x_1, \ldots, x_k$ *of multiplicities* $m_1, \ldots, m_k$ by

$$\mathscr{S}(\mathscr{P}_m; \mathscr{M}; \Delta) = \{s: s_i = s\,|_{I_i} \in \mathscr{P}_m, i = 0, 1, \ldots, k \text{ and } s_{i-1}^{(j)}(x_i-) = s_i^{(j)}(x_i+),$$
$$j = 0, 1, \ldots, m - 1 - m_i \text{ for } i = 1, \ldots, k\}. \tag{2.1}$$

It is well known that $\mathscr{S}(\mathscr{P}_m; \mathscr{M}; \Delta)$ is a linear space of dimension $m + K$, where $K = \sum_1^k m_i$. The elements of this space are piecewise polynomials of order m, and two successive pieces are tied together smoothly at a knot x_i by the continuity of the first $m - 1 - m_i$ derivatives. (If $m_i = m$, an element s of $\mathscr{S}$ need not even be continuous at x_i.)

Before stating our main result on the zeros of elements of $\mathscr{S}$, we need to agree on how to count zeros. First, between any two knots an element $s \in \mathscr{S}$ is a polynomial. Hence, it must either vanish identically, or it can vanish on a finite number of isolated points (at most m). The multiplicities of isolated zeros of s at points between the knots can thus be counted in the usual way. Specifically, we say s has a zero of multiplicity z at a point $t \notin \Delta$ provided

$$s(t) = s'(t) = \cdots = s^{(z-1)}(t) = 0 \neq s^{(z)}(t). \tag{2.2}$$

Next we consider intervals where s vanishes identically. If $[x_i, x_j]$ is such an interval of maximal length, then we count it as a zero of s of multiplicity either m or $m + 1$ according to the following rules:

> If $s(x) = 0$ for $[a, x_j]$ but $s(x) \neq 0$ for $x_j < x < x_j + \epsilon$ and some $\epsilon > 0$, then we count $[a, x_j]$ as an interval of zero of s of multiplicity $z = m$. A similar count is used if s vanishes on an interval ending at b. (2.3)

If s vanishes on an interval interior to (a, b):

> Suppose $s(x) = 0$ for all $[x_i, x_j]$ but $s(x) \neq 0$ for $x_i - \epsilon < x < x_i$ and $x_j < x < x_j + \epsilon$ for sufficiently small $\epsilon > 0$. Then we count the multiplicity of $[x_i, x_j]$ as (2.4)
>
> $$z = \begin{cases} m + 1, & \text{if } m \text{ is even and } s(x_i - \epsilon/2)\, s(x_j + \epsilon/2) < 0, \text{ or} \\ & \text{if } m \text{ is odd and } s(x_i - \epsilon/2)\, s(x_j + \epsilon/2) > 0, \\ m, & \text{otherwise.} \end{cases}$$

This definition is chosen so that s changes sign if z is odd, and does not change sign if z is even. ∎

It remains to consider the case where s is zero at a knot but not in an interval containing the knot, or where s jumps through zero at a knot. If

$t \in \Delta$ and s does not vanish in any interval containing t, we define the multiplicity of the zero t by:

$$\begin{array}{ll}
\multicolumn{2}{l}{\text{Suppose } s(t-) = s'(t-) = \cdots = s^{(l-1)}(t-) = 0 \neq s^{(l)}(t-),} \\
\multicolumn{2}{l}{s(t+) = s'(t+) = \cdots = s^{(r-1)}(t+) \neq s^{(r)}(+). \text{ Let } \alpha = \max(l, r).} \\
\multicolumn{2}{l}{\text{Then we count the multiplicity of } t \text{ as}} \\
z = \alpha + 1, & \text{if } \alpha \text{ is even and } s \text{ changes sign at } t, \text{ or} \\
 & \text{if } \alpha \text{ is odd and } s \text{ does not change sign at } t, \\
\quad = \alpha, & \text{otherwise.}
\end{array} \tag{2.5}$$

This rule counts a jump through 0 at a knot as a zero of multiplicity 1. When s has sufficiently many derivatives at the knot t, then rule (2.5) agrees with the usual rule (2.2).

THEOREM 2.1. *For every $s \in \mathscr{S}(\mathscr{P}_m ; \mathscr{M}; \Delta)$ which is not identically zero,*

$$Z(s) \leqslant m + K - 1, \tag{2.6}$$

where Z counts the number of zeros of s in $[a, b]$, with multiplicities as in (2.2)–(2.5).

Proof. For $m = 1$, Z simply counts the number of times that the piecewise constant function s jumps to or through 0. Since such jumps can only occur at knots, it is clear that $Z(s) \leqslant K = k$ in this case.

We now proceed by induction, using a kind of generalized Rolle's theorem. Suppose the theorem holds for $m - 1$. We show that the existence of a nontrivial $s \in \mathscr{S}(\mathscr{P}_m ; \mathscr{M}; \Delta)$ with $Z(s) \geqslant m + K$ leads to a contradiction. There are two cases. First, suppose that $\mathscr{M}$ is such that no m_i is equal to m. Then s is continuous. We consider the spline Ds defined in each interval I_i as the derivative of the ith (polynomial) component of s. It is clear that Ds is also a polynomial spline, and in fact, $Ds \in \mathscr{S}(\mathscr{P}_{m-1} ; \mathscr{M}'; \Delta)$, where $\mathscr{M}' = (m_1',..., m_k')$ with $m_i' = \min(m_i , m - 1)$.

Examining the definitions of multiple zeros (2.2)–(2.5), we see that if s has a zero of multiplicity $z > 1$ at a point t or on an interval $[x_i , x_j]$, then Ds has a zero of multiplicity $z - 1$ at the same point or on the same interval.

TABLE I

α	s changes sign	$Z(s)$	$\alpha - 1$	Ds changes sign	$Z(Ds)$
even	yes	$\alpha + 1$	odd	no	α
even	no	α	odd	yes	$\alpha - 1$
odd	yes	α	even	no	$\alpha - 1$
odd	no	$\alpha + 1$	even	yes	α

This is well known for the usual definition (2.2). Now for (2.5), for example, we have Table I. A similar table holds in the other cases.

In addition to the zeros of Ds coming from multiple zeros of s, we also observe that by the continuity of s, between any two zeros of s, the spline Ds must have a sign change. Thus, assuming that there are a total of d points and intervals where s vanishes with multiplicities $z_1, \ldots, z_d$ with $Z(s) = \sum_1^d z_i = m + K$, we find that

$$Z(Ds) \geqslant \sum_1^d (z_i - 1) + d - 1 = m + K - 1.$$

As Ds is a spline of order $m - 1$ with at most K knots, this is a contradiction of the induction hypothesis. We conclude that no spline $s \in \mathscr{S}(\mathscr{P}_m; \mathscr{M}; \Delta)$ can exist with $m + K$ zeros.

It remains to consider the case where some of the knots are m-tuple. Suppose for the moment that there is only one such knot, say x_l with $m_l = m$. Define s_R to be the restriction of s to $[x_l, b]$ and s_L to be the restriction of s to $[a, x_l)$. Define $s_L(x_l) = \lim_{x \uparrow x_l} s_L(x)$. Then by what we have already proved, we have

$$Z_{[a,x_l]}(s_L) \leqslant m + \sum_1^{l-1} m_i - 1 \qquad \text{and} \qquad Z_{[x_l,b]}(s_R) \leqslant m + \sum_{l+1}^{k} m_i - 1.$$

In addition, if x_l is a zero of multiplicity z_l for s, then by the definitions of multiplicity, x_l is also a zero of multiplicity $z_l - 1$ of either s_L or s_R. We conclude that

$$Z(s) \leqslant Z_{[a,x_l]}(s_L) + Z_{[x_l,b]}(s_R) + 1 \leqslant m + \sum_1^{l} m_i - 1.$$

If there are several knots of multiplicity m, we simply divide $[a, b]$ into the corresponding number of pieces and argue in the same way. ∎

Each element $s \in \mathscr{S}(\mathscr{P}_m; \mathscr{M}; \Delta)$ has a natural extension to the entire real line $\mathbb{R}$. Indeed, given s as defined in (2.1), we may take $s(x)$ to be the polynomial $s_0(x)$ on all of $(-\infty, x_1)$ and to be the polynomial $s_k(x)$ on all of $[x_k, \infty)$. It is clear that we have not used any properties of $[a, b]$ at all in the above discussion, and hence Theorem 2.1 remains valid if we count zeros as in (2.2)–(2.5) on the entire line.

For simple knots, the bound (2.6) was established by Johnson [6] using Rolle's theorem. Counting only simple zeros but allowing multiple knots, bound (2.6) was proved using Rolle's theorem in my dissertation [19] (see also [20]). Braess [3] noted that the result remains true if nonnodal zeros (cf. [7]) are counted twice. The simple proof using Rolle's theorem given here for the full result combining multiple knots with multiple zeros was made possible by taking a sufficiently strong count of multiple zeros. The counting procedure adopted here was suggested by a similar counting procedure used by Michelli [15] for monosplines (see also Remark 1).

3. Zeros of Periodic Splines

The space of *periodic splines of order m with knots* $x_1, \ldots, x_k$ *of multiplicities* $m_1, \ldots, m_k$ is defined by

$$\mathring{\mathscr{S}}(\mathscr{P}_m; \mathscr{M}; \Delta) = \{s \in \mathscr{S}(\mathscr{P}_m; \mathscr{M}; \Delta): s^{(j)}(b) = s^{(j)}(a), j = 0, 1, \ldots, m-1\}. \tag{3.1}$$

It is not hard to see that this space is a linear space of dimension $K = \sum_1^k m_i$.

In defining the zeros of a spline $s \in \mathring{\mathscr{S}}(\mathscr{P}_m; \mathscr{M}; \Delta)$, it is important to remember that s is periodic. In fact, it is best to think of s as being defined on a circle (obtained by joining the end points a and b of the segment $[a, b]$). Then the points a and b do not play any further role, and the interval $[x_k, x_1)$ is just like any other subinterval defined by Δ. We conclude that the special counting case (2.3) does not arise. Moreover, since our counting procedure is designed so that s changes sign at odd zeros and does not change sign at even ones, it follows that the number of zeros of a periodic spline must always be an even number.

Theorem 3.1. *If* $s \in \mathring{\mathscr{S}}(\mathscr{P}_m; \mathscr{M}; \Delta)$ *is not identically zero, then*

$$\begin{aligned} Z(s) &\leqslant K - 1 \qquad \text{if } K \text{ is odd,} \\ &\leqslant K \qquad\quad\;\; \text{if } K \text{ is even.} \end{aligned}$$

Proof. For $m = 1$ we are dealing with periodic piecewise constant functions whose only possible jump points are at the $x_1, \ldots, x_k$, and the result is easily checked. Now we may proceed by induction on m, using the generalized Rolle's theorem just as in the proof of Theorem 2.1. ∎

4. B-splines

B-splines are of utmost importance in dealing with polynomial splines, and considerable effort has gone into developing their properties. Especially important are the positivity and total positivity properties of certain collocation matrices formed from the B-splines. In this section we show how such results can be obtained immediately from Theorem 2.1.

We begin by recalling the definition of B-splines (cf. [4]). Let $y_{m+1} \leqslant \cdots \leqslant y_{m+K}$ be an enumeration of the sequence $x_1, ..., x_1, ..., x_k, ..., x_k$, where each x_i is repeated exactly m_i times. Let $y_1 \leqslant \cdots \leqslant y_m \leqslant a$ and $b < y_{m+K+1} \leqslant \cdots \leqslant y_{2m+K}$ be arbitrary. For $i = 1, 2, ..., m + K$, we define the B-splines

$$B_i(x) = (-1)^m [y_i, ..., y_{i+m}](x - y)_+^{m-1}, \tag{4.1}$$

where $[y_i, ..., y_{i+m}]$ denotes the divided difference over $y_i, ..., y_{i+m}$. It is well known that the set $\{B_i\}_1^{m+K}$ forms a basis for $\mathscr{S}(\mathscr{P}_m; \mathscr{M}; \Delta)$.

A basic fact about the B-splines is the fact that B_i is positive on (y_i, y_{i+m}) and vanishes outside of $[y_i, y_{i+m}]$. As an illustration of how zero properties can be used, we prove even more in the following theorem.

THEOREM 4.1. *For all* $j = 0, 1, ..., m - 1$,

$$Z_{(y_i, y_{i+m})}(D^j B_i) \leqslant j,$$

where Z *counts multiplicities as in Section* 1. *Moreover, if* $D^j B_i$ *is continuous on* (y_i, y_{i+m}), *then it has precisely* j *distinct zeros there.*

Proof. Since $D^j B_i$ is a polynomial spline of order $m - j$ with at most $m + 1$ knots, counting multiplicities, it follows from Theorem 2.1 that $Z(D^j B_i) \leqslant 2m - j$. But $D^j B_i$ has zeros of multiplicity $m - j$ on each of the intervals $(-\infty, y_i)$ and $[y_{i+m}, \infty)$. We conclude that it can have at most j in (y_i, y_{i+m}).

When $D^j B_i$ is continuous, then a repeated application of the usual Rolle's theorem implies that it must have at least j distinct zeros in (y_i, y_{i+m}). ∎

For simple knots, this result was established already in [4], written in 1946–1947. A similar result on certain kernels, which in fact are polynomial splines, was established even earlier [1] (see Section 7).

In several applications it is necessary to consider matrices formed from the B-splines. Given

$$t_1 \leqslant \cdots \leqslant t_{m+K} = \mathscr{T}_1, ..., \mathscr{T}_1, ..., \mathscr{T}_d, ..., \mathscr{T}_d, \tag{4.2}$$

where each $\mathscr{T}_i$ is repeated exactly $1 \leqslant l_i \leqslant m$ times (so $\sum_1^d l_i = m + K$), we define the matrix

$$M\begin{pmatrix} B_1, ..., B_{m+K} \\ t_1, ..., t_{m+K} \end{pmatrix} = \begin{bmatrix} B_1(\mathscr{T}_1) & \cdots & B_{m+K}(\mathscr{T}_1) \\ DB_1(\mathscr{T}_1) & \cdots & DB_{m+K}(\mathscr{T}_1) \\ \vdots & & \\ D^{l_1-1}B_1(\mathscr{T}_1) & \cdots & D^{l_1-1}B_{m+K}(\mathscr{T}_1) \\ \cdots & & \\ B_1(\mathscr{T}_d) & \cdots & B_{m+K}(\mathscr{T}_d) \\ \vdots & & \\ D^{l_d-1}B_1(\mathscr{T}_d) & \cdots & D^{l_d-1}B_{m+K}(\mathscr{T}_d) \end{bmatrix}. \tag{4.3}$$

We denote the determinant of this matrix by

$$D\begin{pmatrix} B_1, \ldots, B_{m+K} \\ t_1, \ldots, t_{m+K} \end{pmatrix}.$$

The following result gives the precise conditions on the t's which assure that D is not 0.

THEOREM 4.2. *Let $m > 1$ and suppose $\{t_i\}_1^{m+K}$ as in* (4.2) *with at most m t's and x's equal to any one value. Then*

$$D\begin{pmatrix} B_1, \ldots, B_{m+K} \\ t_1, \ldots, t_{m+K} \end{pmatrix} \neq 0$$

if and only if

$$y_i < t_i < y_{i+m}, \qquad i = 1, 2, \ldots, m + K. \tag{4.4}$$

Proof. It is easily seen using Laplace's expansion that if (4.4) fails then $D = 0$ (cf. [7, 13]). Now suppose that (4.4) holds, but that the determinant is 0. Then there exists a nontrivial linear combination of the B's, say $s(x) = \sum_1^{m+K} c_i B_i(x)$ which vanishes at all of the t's. Let c_l be the first nonzero coefficient, and suppose $l \leqslant r \leqslant m + K$ is the smallest index so that s vanishes on an interval with left endpoint y_r. Then $\tilde{s} = \sum_l^r c_i B_i$ has m-tuple zeros on $(-\infty, y_l)$ and on $[y_r, \infty)$. In addition, it vanishes at the points $t_l, \ldots, t_r$ which lie in (y_l, y_{r+m}) by (4.4). This is a total of $2m + r - l + 1$ zeros. But $\tilde{s}$ only has $m + r - l + 1$ knots, and this is a contradiction of Theorem 2.1. The determinant cannot be zero when (4.4) holds. ∎

The conditions (4.4) require that each t_i be in the interior of the support of the corresponding B-spline B_i, $i = 1, 2, \ldots, m + K$. Theorem 4.1 can be improved to determine the sign.

COROLLARY 4.3. *Under conditions* (4.4) *the determinant D in Theorem* 4.2 *is strictly positive.*

Proof. One shows that D is positive for some choice of t's satisfying (4.4), and then uses the continuity of D to deduce the positivity for all such t. As the details have nothing to do with zeros of splines, we do not bother with them. ∎

Theorem 4.2 and Corollary 4.3 remain valid for $m = 1$ if in (4.4) equality is permitted on the left-hand side. These two results were first established by Karlin [9] with distinct t's and x's. DeBoor [2] allowed multiple knots as well as multiple t's. Using the purely algebraic method of deBoor [2] (cf. also the proof of Theorem 6.3), we can also prove the following result of deBoor.

COROLLARY 4.4. *Let $m > 1$. Suppose $t_1 \leqslant \cdots \leqslant t_p$ with at most m t's and x's equal to any one value. Suppose that $1 \leqslant \nu_1 < \cdots < \nu_p \leqslant m + K$. Then*

$$D\begin{pmatrix} B_{\nu_1}, \ldots, B_{\nu_p} \\ t_1, \ldots, t_p \end{pmatrix} \geqslant 0.$$

Strict positivity holds if and only if

$$t_i \in (y_{\nu_i}, y_{\nu_i+m}), \qquad i = 1, 2, \ldots, p.$$

For a sequence of distinct $t_1 < \cdots < t_p$, the first part of Corollary 4.4 was proved by Karlin [7] by using results on a certain Green's function (cf. Section 6).

5. PERIODIC B-SPLINES

In this section we consider properties of a matrix formed from a basis of B-splines for the space $\mathring{\mathscr{S}}(\mathscr{P}_m; \mathscr{M}; \Delta)$ defined in (3.1). To define the B-splines, let $y_{m+1} \leqslant \cdots \leqslant y_{m+K}$ be a renumbering of the set $x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k$ with each x_i repeated exactly m_i times. Now define $y_i = y_{i+K} - (b - a)$, $i = 1, 2, \ldots, m$. Let $\{B_i\}_1^K$ be the corresponding B-splines, and define

$$\mathring{B}_i(x) = B_i(x), \qquad i = m + 1, \ldots, K \tag{5.1}$$

and

$$\begin{aligned} \mathring{B}_i(x) &= B_i(x - (b - a)), && y_{K+1} \leqslant x \leqslant b \\ &= B_i(x), && a \leqslant x \leqslant y_{K+1} \end{aligned} \qquad i = 1, 2, \ldots, m. \tag{5.2}$$

It is easily seen that these definitions result in periodic splines, and that the set $\{\mathring{B}_i\}_1^K$ forms a basis for $\mathring{\mathscr{S}}(\mathscr{P}_m; \mathscr{M}; \Delta)$. Given $t_1 \leqslant \cdots \leqslant t_K$, we are interested in properties of the matrix

$$M\begin{pmatrix} \mathring{B}_1, \ldots, \mathring{B}_k \\ t_1, \ldots, t_K \end{pmatrix},$$

where equalities among the t's implies that derivatives are to be introduced just as in (4.3).

It is important to note that in the periodic case where the $t_1 \leqslant \cdots \leqslant t_K$ lie on a circle, there is no natural way to number the t's; that is, any t can be considered as t_1, and moreover, t_1 is to be thought of as following t_K. When K is *odd* the numbering of the t's has no effect on the value of the determinant as the number of interchanges of rows required to change

$$M\begin{pmatrix} \mathring{B}_1, ..., \mathring{B}_K \\ t_1, ..., t_K \end{pmatrix} \quad \text{into} \quad M\begin{pmatrix} \mathring{B}_1, ..., \mathring{B}_K \\ t_2, ..., t_K, t_1 \end{pmatrix}$$

is even and the value of the determinant is the same. This suggests that we consider only K odd here (see also Example 5.2).

THEOREM 5.1. *Let K be odd. Let $m > 1$, and suppose $t_1 \leqslant \cdots \leqslant t_K$ are any points in $[a, b]$ with at most m t's and x's equal to any one value. Then*

$$D\begin{pmatrix} \mathring{B}_1, ..., \mathring{B}_K \\ t_1, ..., t_K \end{pmatrix} \neq 0 \tag{5.3}$$

if and only if, for some integer $0 \leqslant p \leqslant K - 1$,

$$t_{i+p} \in (y_i, y_{i+m}), \qquad i = 1, 2, ..., K. \tag{5.4}$$

(Here the y's are to be thought of as points on the circle). Moreover, if (5.4) holds, then D is actually positive.

Proof. First we show that D is not zero whenever (5.4) holds. Suppose to the contrary that (5.4) holds but $D = 0$. Then there exists a nontrivial $s = \sum_1^k c_i\mathring{B}_i$ with zeros at all of the $t_1, ..., t_K$, counting multiplicities. If none of the c's are zero, then by the local linear independence of the B-splines, s cannot vanish on any subinterval of the circle. This is a contradiction of Theorem 3.1, as s has K zeros counting multiplicities, but only K (odd) knots. (The fact that s is not identically zero on any subinterval assures that no zeros are swallowed up in an interval of identical vanishing.) On the other hand, if s does vanish on an interval, then there exist l and r such that $\tilde{s} = \sum_l^r c_iB_i$ vanishes on an interval to the left of y_l and to the right of y_{r+m}, but on no subinterval of (y_l, y_{r+m}). Since (5.4) implies $y_l < t_{l+p} < t_{r+p} < y_{r+m}$, we conclude $\tilde{s}$ has a total of $2m + r - l + 1$ zeros, but only $m + r - l$ knots. This is a contradiction of Theorem 2.1. We conclude that (5.4) implies $D \neq 0$.

The fact that D is actually positive under the hypothesis (5.4) follows from a continuity argument just as in the proof of Corollary 4.3.

It remains to show that when (5.4) fails, $D = 0$. Because of the cyclic nature, this is not such a trivial exercise as in the nonperiodic case. To prove this, we observe that (5.4) is equivalent to the statement

For each $j = 1, 2,..., K$, the support set of any j consecutive B-splines contains at least j points from the set $\{t_i\}_1^K$. (5.5)

Leaving the proof of this equivalence aside for the moment, we now see that if (5.4) fails, then for some $1 \leqslant j \leqslant K$, there exist some j consecutive B-splines whose support contains only $j - 1$ of the t's. But then the matrix M has some j consecutive columns which have nonzero entries in at most $j - 1$ rows. Expanding by Laplace's expansion yields the value 0 for the determinant.

It is easy to see that (5.4) implies (5.5). Indeed, by (5.4) we can associate a distinct t with each B-spline, lying in its support But then any two consecutive B-splines have two t's in their support, etc. The converse is somewhat more difficult. We suppose now that (5.5) holds, and prove that there exists a p satisfying (5.4).

For convenience, let U_i denote the set (y_i, y_{i+m}). Suppose that for any numbering of the t's with $t_1 \in U_1$, that for some $L < K$, $t_j \in U_j$, $j = 1, 2,..., L$, but $t_{L+1} \notin U_{L+1}$. We show that this is a contradiction, so that some numbering must work. In particular, if $\{t_i\}_1^K$ is such a set with $L < K$, we show how to construct another numbering with more than L good t's.

There are two cases: Either $t_{L+1} \geqslant y_{L+m+1}$, or $t_{L+1} \leqslant y_{L+1}$. We consider the first case. Define $\tilde{t}_j = t_{j-1}$, $j = 1,..., K$ (where $t_0 = t_K$). We claim that $\tilde{t}_j \in U_j$ for $j = 1, 2,..., L + 1$. Indeed, $\tilde{t}_j = t_{j-1} < y_{m+j-1} \leqslant y_{m+j}$, $j = 2,..., L + 1$. Moreover, $\tilde{t}_1 < t_1 < y_{m+1}$. On the other hand, if $\tilde{t}_\nu = t_{\nu-1} \leqslant y_\nu$ for some $1 \leqslant \nu \leqslant L + 1$, then (y_ν, y_{L+m+1}) contains only the points $t_\nu,..., t_L$. This is not enough to satisfy (5.5).

Finally, we consider the second case, $t_{L+1} \leqslant y_{L+1}$. Suppose q is such that $t_{L+1} \leqslant \cdots \leqslant t_{L+q} \leqslant y_{L+1} < t_{L+q+1}$. Now we define $\tilde{t}_j = t_{j+q}$, $j = 1, 2,..., K$ (with $t_{K+i} = t_i$, $i = 1,...,$). Again, we claim that $\tilde{t}_j \in U_j$, $j = 1, 2,..., L + 1$. Indeed, $\tilde{t}_j = t_{j+q} \geqslant t_j > y_j$, $j = 1, 2,..., L$. Also, $\tilde{t}_{L+1} = t_{L+q+1} > y_{L+1}$. On the other hand, $\tilde{t}_{L+1} = t_{L+q+1} < y_{L+m+1}$, for otherwise the interval (y_{L+1}, y_{L+m+1}) contains no t's. Moreover, $\tilde{t}_{L+1-m} \leqslant \cdots \leqslant \tilde{t}_L \leqslant y_{L+1} \leqslant \cdots \leqslant y_{L+m}$. Hence if $L < m$ we have proved $t_j < y_{m+j}$, $j = 1,..., L$. Finally, if $L \geqslant m$, and if for some $1 \leqslant \nu \leqslant L + 1 - m$ we have $\tilde{t}_\nu \geqslant y_{\nu+m}$, then the interval $[y_{\nu+m}, y_{L+1}]$ contains $t_{\nu+q},..., t_{L+q}$. In this case the complement doesn't contain enough t's to fulfill (5.5). ∎

The fact that we used Theorem 3.1 in the proof of Theorem 5.1 also indicates that we cannot expect it to be true for K even (since then $Z(s) \leqslant K$ is all that Theorem 3.1 asserts). The following example shows that indeed the conditions (5.4) are not sufficient for the nonsingularity of M in the even case.

EXAMPLE 5.2. Let $[a, b] = [0, 2]$ and choose $\Delta = \{\frac{1}{2}, \frac{3}{2}\}$. Consider $m = 2$ so that we are working with piecewise linear splines. Here $K = 2$, and the two periodic B-splines forming a basis for $\mathscr{S}_2(\Delta)$ are easily constructed. Now if we choose $t_1 = 0$ and $t_2 = 1$, then

$$M\begin{pmatrix} \mathring{B}_1, \mathring{B}_2 \\ t_1, t_2 \end{pmatrix} = 0,$$

although (5.4) is satisfied (compare Fig. 1).

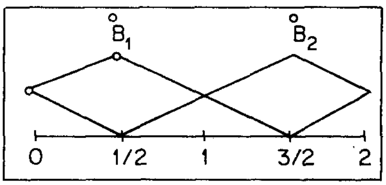

FIG. 1. A basis for $\mathscr{S}_2(\{\frac{1}{2}, \frac{3}{2}\})$.

Although these statements in the periodic case are quite natural, I have not been able to find them in the literature. It is somewhat surprising that the proof of the necessity of (5.4) turns out to be so delicate. For some results on matrices formed from the Green's function of the following section with periodic boundary conditions, see [10].

6. A GREEN'S FUNCTION

As another application of the zero properties of Section 1, we establish some total positivity properties of the kernel

$$g(t, y) = (t - y)_+^{m-1} \tag{6.1}$$

(g is the Green's function for D^m with initial conditions $D^j f(a) = 0$, $j = 0, 1,..., m - 1$).

In general, if $G(t, y)$ is a kernel defined on a square $[a, b] \times [a, b]$, then, given

$$\begin{aligned} y_1 &\leqslant \cdots \leqslant y_p = x_1, ..., x_1, ..., x_k, ..., x_k, \\ t_1 &\leqslant \cdots \leqslant t_p = \mathscr{T}_1, ..., \mathscr{T}_1, ..., \mathscr{T}_d, ..., \mathscr{T}_d \end{aligned} \tag{6.2}$$

(where each x_i is repeated exactly m_i times with $\sum_1^k m_i = p$ and each $\mathscr{T}_i$ is repeated exactly l_i times with $\sum_1^d l_i = p$), we define the matrix

$$G\begin{pmatrix} y_1, \dots, y_p \\ t_1, \dots, t_p \end{pmatrix} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1k} \\ G_{21} & G_{22} & \cdots & G_{2k} \\ \cdots \\ G_{d1} & G_{d2} & \cdots & G_{dk} \end{bmatrix}, \tag{6.3}$$

where

$$G_{ij} = \begin{bmatrix} G(t_i, y_j) & D_y G(t_i, y_j) & \cdots & D_y^{l_j-1} G(t_i, y_j) \\ D_t G(t_i, y_j) & D_t D_y G(t_i, y_j) & \cdots & D_t D_y^{l_j-1} G(t_i, y_j) \\ \cdots \\ D_t^{m_i-1} G(t_i, y_j) & D_t^{m_i-1} D_y G(t_i, y_j) & \cdots & D_t^{m_i-1} D_y^{l_j-1} G(t_i, y_j) \end{bmatrix} \tag{6.4}$$

(cf. [7]). This matrix is defined whenever G has mixed derivatives of the required order.

If we take $D_y(t - y)_+^j = -j(t - y)_+^{j-1}$ for all $1 \leqslant j \leqslant m - 1$ and take D_t always to be the right derivative, then the corresponding matrix for the kernel g is defined for all y's and t's as in (6.2) as long as $1 \leqslant m_i$, $l_i \leqslant m$.

THEOREM 6.1. *Let $m > 1$. Suppose in* (6.2) *that $1 \leqslant m_i$, $l_j \leqslant m$, for all i and j. Then if at most m t's and y's are equal to any one value,*

$$\det g\begin{pmatrix} y_1, \dots, y_p \\ t_1, \dots, t_p \end{pmatrix} \geqslant 0. \tag{6.5}$$

Moroever, this determinant is strictly positive if and only if

$$t_{i-m} < y_i < t_i, \qquad i = 1, 2, \dots, p, \tag{6.6}$$

where the left-hand inequality is ignored if $i < m$.

Proof. It is not hard to show via Laplace's expansion that the determinant is zero whenever (6.6) fails (cf. [7, 13]). Now suppose that (6.6) holds, but that the determinant is 0. Then there exist coefficients, not all zero, so that

$$s(t) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij}(t - x_i)_+^{m-j}$$

is zero at each of the points $t_1 \leqslant \cdots \leqslant t_p$, along with appropriate right derivatives in case of multiplicities. Suppose l is the maximal index such that s vanishes identically up to y_l.

If s does not vanish on any interval to the right of y_l, then it is a nontrivial spline with $p - l + 1$ knots which vanishes (counting multiplicities) a total of $m + p - l + 1$ times (m times to the left of y_l and at the $y_l < t_l \leqslant \cdots \leqslant t_p$; these points cannot be swallowed up in an interval where s vanishes identically). This contradicts Theorem 2.1.

On the other hand, if s vanishes somewhere to the right, there exists an r such that it does not vanish identically on any subinterval of $(y_l, y_{r+m}) = (x_\alpha, x_\beta)$, but does vanish identically on an interval with left endpoint y_{r+m}. This means that the spline $\tilde{s} = \sum_{i=\alpha}^{\beta} \sum_{j=1}^{m_i} c_{ij}(t - x_i)_+^{m-j}$ vanishes m times to the left, m times to the right, and at the points $t_l, \ldots, t_r$, which, by (6.6), lie in (y_l, y_{r+m}). As $\tilde{s}$ only has $m + r - l + 1$ knots, these $2m + r - l + 1$ zeros contradict Theorem 2.1. We conclude that the determinant cannot be zero when (6.6) holds.

The fact that the determinant is actually positive under (6.6) follows from a continuity argument, as in the proof of Corollary 4.2. ∎

The earliest version of Theorem 6.1 is due to Schoenberg and Whitney [18], where distinct sequences $y_1 < \cdots < y_p$ and $t_1 < \cdots < t_p$ were considered. Their proof relied on techniques from Fourier analysis. A direct theorem allowing multiple t's but only simple y's was established in [19]. The complete theorem was proved by a complicated multiple induction by Karlin and Ziegler [13] (see also [7]). The proof given here is much simpler and is totally different.

Theorem 6.1 can be used to prove strong determinantal properties of another basis for $\mathscr{S}(\mathscr{P}_m; \mathscr{M}; \Delta)$. We define

$$\tilde{B}_1(t), \ldots, \tilde{B}_{m+K}(t) = (t - a)^{m-1}, \ldots, 1, (t - x_1)_+^{m-1}, \ldots, (t - x_1)_+^{m-m_1}, \ldots,$$
$$(t - x_k)_+^{m-1}, \ldots, (t - x_k)_+^{m-m_k}. \tag{6.7}$$

It is well known that this is a basis for $\mathscr{S}(\mathscr{P}_m; \mathscr{M}; \Delta)$.

THEOREM 6.2. *Let $m > 1$. Suppose $a \leqslant t_1 \leqslant \cdots \leqslant t_{m+K} \leqslant b$ are real numbers with at most m t's and x's equal to any one value. Then* (*cf.* (4.3))

$$D\begin{pmatrix} \tilde{B}_1, \ldots, \tilde{B}_{m+K} \\ t_1, \ldots, t_{m+K} \end{pmatrix} \geqslant 0. \tag{6.8}$$

Moreover, strict inequality holds if and only if

$$y_i < t_i < y_{i+m}, \qquad i = 1, 2, \ldots, m + K, \tag{6.9}$$

where

$$y_1 \leqslant \cdots \leqslant y_{m+K} = x_0, \ldots, x_0, \ldots, x_k, \ldots, x_k, \tag{6.10}$$

and where each x_i is repeated exactly m_i times ($m_0 = m$). The points $y_{m+K+1} \leqslant \cdots \leqslant y_{2m+K}$ are arbitrary points $> b$.

Proof. We may apply Theorem 6.1 with $p = m + K$. ▮

We conclude by establishing a total positivity result for the matrix formed from the $\tilde{B}$'s.

THEOREM 6.3. *Let $\{\tilde{B}_i\}_1^{m+K}$ and $\{y_i\}_1^{2m+K}$ be as in Theorem 6.1. Let $1 \leqslant \nu_1 < \cdots < \nu_p \leqslant m + K$, and suppose $a \leqslant t_1 \leqslant \cdots \leqslant t_p \leqslant b$ is such that no more than m t's and y's have any one value. Then*

$$D\begin{pmatrix} \tilde{B}_{\nu_1}, \ldots, \tilde{B}_{\nu_p} \\ t_1, \ldots, t_p \end{pmatrix} \geqslant 0. \tag{6.11}$$

The determinant is strictly positive if and only if

$$t_i \in (y_{\nu_i}, y_{\nu_i+m}), \qquad i = 1, 2, \ldots, p. \tag{6.12}$$

Proof. The fact that the determinant is 0 if (6.12) fails is established directly with the help of Laplace's expansion. Assuming (6.12) holds, the positivity of D is now established by induction on p and q = number of gaps in the sequence $\nu_1, \ldots, \nu_p$. When $q = 0$, this is just Theorem 6.2. Suppose now the assertion holds for q, and all determinants of size $p - 1$.

First, we observe that there is no loss of generality in assuming that $y_{\nu_i+1} < t_i < y_{\nu_i+m-1}$, $i = 1, 2, \ldots, p$. Indeed, if this fails, say $t_j \leqslant y_{\nu_j+1}$ for some $1 \leqslant j \leqslant p$, then the entries in the first j rows and columns $j + 1$ through p of M are all 0 since $t_1 \leqslant \cdots \leqslant t_j \leqslant y_{\nu_j+1} \leqslant \cdots \leqslant y_{\nu_p}$. Then D can be written as the product of two smaller determinants.

Now suppose that i is one of the missing indices in the sequence $\nu_1, \ldots, \nu_p$, and that l is such that $\nu_1 < \cdots < \nu_l < i < \nu_{l+1} < \cdots < \nu_p$. By a determinantal identity (cf. [7, p. 507; 2]):

$$\begin{aligned} &D\begin{pmatrix} \tilde{B}_{\nu_2}, \ldots, \tilde{B}_{\nu_l}, \tilde{B}_i, \tilde{B}_{\nu_{l+1}}, \ldots, \tilde{B}_{\nu_{p-1}} \\ t_1, \ldots, t_{l-1}, t_l, t_{l+1}, \ldots, t_{p-1} \end{pmatrix} D\begin{pmatrix} \tilde{B}_{\nu_1}, \ldots, \tilde{B}_{\nu_p} \\ t_1, \ldots, t_p \end{pmatrix} \\ &\quad = D\begin{pmatrix} \tilde{B}_{\nu_2}, \ldots, \tilde{B}_{\nu_p} \\ t_1, \ldots, t_{p-1} \end{pmatrix} D\begin{pmatrix} \tilde{B}_{\nu_1}, \ldots, \tilde{B}_{\nu_l}, \tilde{B}_i, \tilde{B}_{\nu_{l+1}}, \ldots, \tilde{B}_{\nu_{p-1}} \\ t_1, \ldots, t_l, t_{l+1}, t_{l+2}, \ldots, t_{p-1} \end{pmatrix} \\ &\qquad + D\begin{pmatrix} \tilde{B}_{\nu_1}, \ldots, \tilde{B}_{\nu_{p-1}} \\ t_1, \ldots, t_{p-1} \end{pmatrix} D\begin{pmatrix} \tilde{B}_{\nu_2}, \ldots, \tilde{B}_{\nu_l}, \tilde{B}_i, \tilde{B}_{\nu_{l+1}}, \ldots, \tilde{B}_{\nu_p} \\ t_1, \ldots, t_l, t_{l+1}, t_{l+2}, \ldots, t_p \end{pmatrix}. \end{aligned}$$

Since the sequence $\nu_1, \ldots, \nu_l, i, \nu_{l+1}, \ldots, \nu_p$ has at most q gaps, and since each of the t's in the above determinants lie in the proper intervals by the assumption we were permitted above, we conclude that each of the determinants on the right-hand side is positive, and the determinant in front of the desired determinant on the left is also positive. The result follows. ▌

Theorem 6.2 was first proved by Karlin and Schumaker [12] by entirely different methods. Theorem 6.3 was obtained by using a smoothing method. The direct proof given here is based on ideas of deBoor [2], used in the B-spline case. For some similar results on determinants involving the $\{\tilde{B}_{ij}\}$, but with additional boundary conditions, see [9].

7. Zeros of g-splines

In this section we consider a larger class of polynomial splines than that defined in Section 2. Let $E = (E_{ij})_{j=0,i=1}^{m-1,k}$ be a matrix consisting of 0's and 1's. Suppose that the number of entries equal to 1 is equal to K. We define

$$\mathscr{S}(\mathscr{P}_m\,;E;\Delta) = \{s\colon s\,|_{I_i} \in \mathscr{P}_m\,,\ i = 0, 1,\ldots, k \text{ and}$$
$$s^{(m-j-1)}(x_i-) = s^{(m-j-1)}(x_i+),\ \text{all } i, j \text{ with } E_{ij} = 0\}. \tag{7.1}$$

We call the space defined in (7.1) the space of *g-splines of order m corresponding to the incidence matrix E*. It is not hard to show this space is of dimension $m + K$. The space of Section 2 corresponds to the Hermite case where $E_{ij} = 1$ implies $E_{i\nu} = 1$ for all $\nu = 0, 1,\ldots, j$.

Zero properties for the space of g-splines are somewhat more difficult to establish as they depend heavily on the structure of E. Suppose we define $\tilde{Z}(s)$ for an element $s \in \mathscr{S}(\mathscr{P}_m\,; E; \Delta)$ by using the usual count (2.2) at a point $t \notin \Delta$, by counting zero intervals as $z = 1$, and by taking $z = \min(l, r)$ (cf. (2.5)) in case $t \in \Delta$. This is a weaker counting method than that used in Section 3. Now with these definitions, it is easy to carry over the methods of Lorentz [14] to show

THEOREM 7.1. *Let p be the number of sequences of consecutive* 1*'s in E with an odd number in them, and which do not start in the first column. Then for any $s \in \mathscr{S}(\mathscr{P}_m\,; E; \Delta)$ which is not identically* 0,

$$\tilde{Z}_{(x_1,x_k)}(s) \leqslant m + K + p - 1.$$

This result can also be stated for splines vanishing outside an interval. The following result is due to Lorentz [14] (see also [5] for the simple zero case).

THEOREM 7.2. *Suppose $s \in \mathcal{S}(\mathcal{P}_m; E; \Delta)$ is not identically zero, but that it vanishes outside of (x_1, x_k). Then*

$$Z(s) \leqslant K + p' - 1 - m',$$

where m' is the exact *order of s and where p' is the number of odd supported sequences of E (defined by the property that if the sequence begins in column $j > 1$ and in row i, then there must exist rows $i' < i < i''$ and columns $0 \leqslant j', j'' < j$ such that $E_{i'j'} = E_{i''j''} = 1$).*

It does not appear that these results can be strengthened to permit the use of Z defined in Section 3 without increasing the bound by some further constant depending on E. We have the following example.

EXAMPLE 7.3. Let $m = 9$ and choose $\Delta = \{0, 1\}$. Let

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now the spline

$$\begin{aligned} s(t) &= -t^8, & t &< 0, \\ &= 1, & 0 &\leqslant t < 1, \\ &= -(t-1)^8, & 1 &\leqslant t, \end{aligned}$$

belongs to $\mathcal{S}(\mathcal{P}_m; E; \Delta)$. If we use the Z of Section 2, then this spline has 18 zeros (9 at each of the points 0 and 1), but Theorem 7.1 requires $Z(s) \leqslant 14$ ($m = 9, K = 4, p = 2$).

8. REMARKS

(1) There has probably been more effort expended on the zero properties of monosplines than on splines themselves, perhaps because monosplines are connected with quadrature formulas. For results on the zeros of monosplines with simple knots but counting multiple zeros, see [16, 6, 12]. For monosplines with boundary conditions, see [8, 11]. The result for monosplines with multiplicities of both knots and zeros was obtained by Michelli [15]. Although his counting procedure is identical with ours, his proof of a bound on the zeros of monosplines was based on the Budan–Fourier theorem on the sign changes of polynomials.

(2) There are versions of all of the results of this paper for Tchebycheffian splines (cf. [7, 13] for a definition). In fact, the method of proof used to obtain zero properties here can be used on a generalized class of

Tchebycheffian splines where it is not assumed that the underlying system is an ECT system. This permits the construction of positive local support bases for certain classes of singular splines. We treat these ideas in another paper.

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